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The drag of a body moving transversely in a confined stratified fluid

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The slow motion of a body through a stratified fluid bounded laterally by insulating walls is studied for both large and small Peclet number. The Taylor column and its associated boundary and shear layers are very different from the analogous problem in a rotating fluid. In particular, the large Peclet number problem is non-linear and exhibits mixing of statically unstable fluid layers, and hence the drag is order one; whereas the small Peclet number flow is everywhere stable, and the drag is of the order of the Peclet number.

1. Introduction

The purpose of this paper is to study the slow motion, in the sense of small Froude number, of a two-dimensional body through a stratified fluid bounded by two parallel, vertical walls. We will consider only horizontal motion at speed U. For the most part, our attention will be confined to a thin flat plate of height $2h$, beginning its motion at distances L_1 and L_2 from the right and left walls respectively.

One might expect the flow resulting from such a motion to be analogous to the rise of a body through a rotating fluid (Moore & Saffman 1968). Veronis (1967) has shown that such an analogy can be expected only if the Prandtl number, σ, of the stratified flow is of order unity, *and* if the boundary conditions are on the thermal field, and of a particular form. The boundary conditions on the plate are kinematic, not thermal, so that no analogy can be expected even if $\sigma = O(1)$.

Throughout this work, we use the Boussinesq equations

$$\nabla . \mathbf{u} = 0, \tag{1.1}$$

$$D\rho/Dt = \kappa \nabla^2 \rho, \tag{1.2}$$

$$D\mathbf{u}/Dt + \nabla p = -\rho g \mathbf{k} + \nu \nabla^2 \mathbf{u}, \tag{1.3}$$

where the reference density is unity. The boundary conditions are, if (x, z) are the horizontal and vertical co-ordinates,

$$\mathbf{u} = 0, \quad \partial\rho/\partial x = 0 \quad \text{on} \quad x = L_1, \quad x = -L_2, \tag{1.4}$$

and

$$\mathbf{u} = U\mathbf{i}, \quad \mathbf{n}.\nabla\rho = 0 \quad \text{on the body}, \tag{1.5}$$

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where $\mathbf{n}$ is the outward normal on the body surface. In general, there is also the initial condition

$$\rho = S(z), \quad S'(z) < 0 \quad \text{at} \quad t = 0. \tag{1.6}$$

In most of this paper, we consider the special case

$$S(z) = 1 - \beta z \quad (\beta > 0). \tag{1.7}$$

The ratio of the orders of magnitude of the left- and right-hand sides of (1.2) is the Peclet number,

$$Pe = Uh^2/\kappa L, \tag{1.8}$$

where L is a characteristic horizontal dimension. The smallness of the Froude number

$$\epsilon = U/(g\beta h^2)^{\frac{1}{2}} \tag{1.9}$$

does not impose any restrictions on Pe, and it is clear that the structure of the flow may be quite different depending on $O(Pe)$ and $O(\sigma)$.

We will investigate in detail solutions for both $Pe \ll 1$ and $Pe \gg 1$. In all cases it will be supposed that viscosity is sufficiently small for viscous effects to be confined to thin boundary and shear layers.

2. Solution for vanishing diffusion

We will suppose, for the present, that the fluid has zero coefficient of diffusion, which means that density discontinuities are allowable in the solution. The complete equations, in that case, are (1.1), (1.3), and (1.2) with $\kappa \equiv 0$, which may be written in non-dimensional form, after subtracting a uniform hydrostatic pressure gradient, as

$$\nabla' . \mathbf{u}' = 0, \tag{2.1}$$

$$D\rho'/Dt' = 0, \tag{2.2}$$

$$\epsilon^2 D\mathbf{u}'/Dt' + \nabla' p' = -\rho'\mathbf{k} + A\nabla'^2\mathbf{u}'. \tag{2.3}$$

The solution is to be studied for $\epsilon \ll 1, A \ll 1$, where U and h are the characteristic velocity and length, respectively, and

$$A \equiv U\nu/g\beta h^3 = \epsilon^2 \nu/Uh. \tag{2.4}$$

3. The interior solution for $\kappa \equiv 0$

Denoting the region $Ut < x < L_1$, $|z| < h$ by $(+)$ and the region $-L_2 < x < Ut$ by $(-)$ (cf. figure 1), we seek a solution to

$$\nabla . \mathbf{u} = 0, \tag{3.1a}$$

$$D\rho/Dt = 0, \tag{3.1b}$$

$$\nabla p + \rho g \mathbf{k} = 0, \tag{3.1c}$$

subject to modified (inviscid) boundary conditions

$$u = 0 \quad \text{on} \quad x = L_1, \quad x = -L_2, \tag{3.1d}$$

$$u = U \quad \text{on} \quad x = Ut, \quad |z| < h \tag{3.1e}$$

and the initial condition

$$\rho = 1 - \beta z \quad \text{on} \quad t = 0. \tag{3.1f}$$

So (3.1) constitutes the problem to be solved when the body is a plate $x = Ut$, $|z| < h$. Hydrostatic equilibrium (cf. (3.1c)) means that $\rho = \rho(z,t)$, so ρ = constant on a pathline implies from (3.1b) that $w = w(z,t)$ as well. Therefore, (3.1a) integrates to

$$u(x,z,t) = f(z,t) - x\,\partial w(z,t)/\partial z,$$

which is a result independent of boundary and initial conditions. Using (3.1d) and (3.1e), it is easily deduced that

$$w_\pm = \pm Uz/l_{\substack{1\\2}} \quad \text{in } (\pm), \tag{3.2}$$

$$w \equiv 0, \quad |z| > h, \tag{3.3}$$

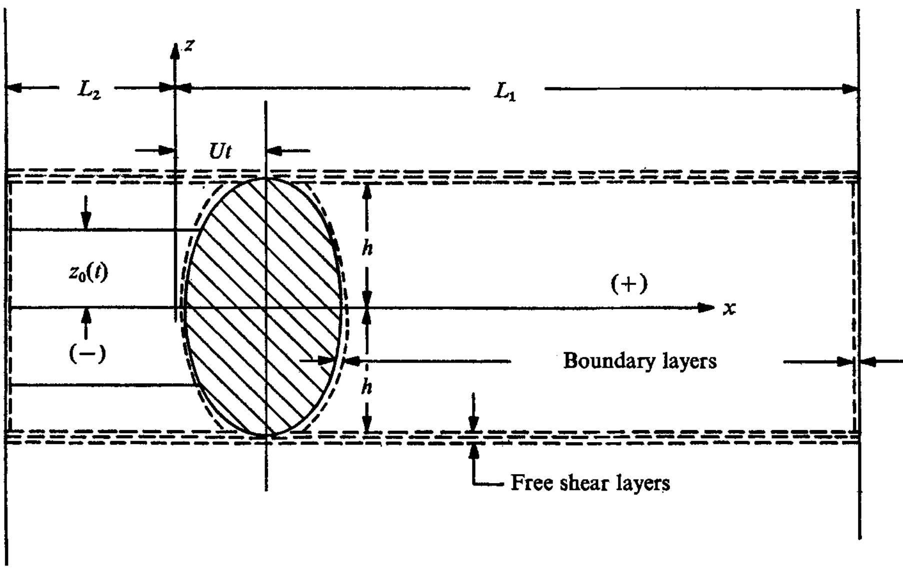

FIGURE 1. Flow structure and co-ordinate system for body moving to the right at speed U.

where $l_1 = L_1 - Ut$, $l_2 = L_2 + Ut$. Then (3.1b) takes the form

$$\frac{\partial\rho}{\partial t} \pm \frac{Uz}{l_{\substack{1\\2}}}\frac{\partial\rho}{\partial z} = 0 \quad \text{in } (\pm) \tag{3.4}$$

and simply ρ = const. for $|z| > h$. The (+) problem is an initial-value problem, and the solution, using (3.1f), is

$$\rho_+ = 1 - \beta z l_1/L_1. \tag{3.5a}$$

In (−), however, the solution to (3.4) requires data on both $|z| < h$, $t = 0$ and on $|z| = h$, $t > 0$, since the characteristics, given by

$$dz/dt = -Uz/l_2,$$

slope inward toward smaller $|z|$ for $t > 0$, and those for $|z|$ greater than $z_0 = hL_2/l_2$ meet $|z| = h$. For $|z| < z_0$, the solution is determined by the initial values and (3.4) gives

$$\rho_- = 1 - \beta z l_2/L_2, \quad |z| < z_0. \tag{3.5b}$$

Equation (3.2) implies that fluid leaving $(+)$ enters $(-)$, and therefore

$$\rho_+(\pm h) = \rho_-(\pm h)$$

gives the boundary condition on $|z| = h$.

So, solving (3.4) for ρ_- in $z_0 < |z| < h$ subject to

$$\rho_-(\pm h) = 1 \mp \beta h l_1/L_1$$

and denoting ρ_- in $z_0 < |z| < h$ by $\hat{\rho}_-$, we get

$$\hat{\rho}_- = 1 - \beta h(L_1 + L_2)/L_1 + \beta z l_2/L_1 \quad (z_0 < z < h); \tag{3.5c}$$

$$\hat{\rho}_- = 1 + \beta h(L_1 + L_2)/L_1 + \beta z l_2/L_1 \quad (-h < z < -z_0). \tag{3.5d}$$

We now notice that $\qquad \partial\hat{\rho}_-/\partial z = \beta l_2/L_2 > 0,$

so the layers of fluid in $z_0 < |z| < h$, $x < Ut$ are statically unstable. The solution is therefore physically unrealistic in this region and some mixing *will* take place. We consider below the probable form of the density gradient in this part of the flow. Everywhere else, the density gradient given by the solution (3.5) is negative.

It is obvious that the density distribution depends only on the distance moved and is independent of the velocity of the body; i.e. for arbitrary $U(t)$ it is sufficient to replace Ut by $\int_0^t U\,dt$.

The case when the body is represented by

$$x - Ut = F_+(z) \quad \text{in } (+),$$
$$x - Ut = -F_-(z) \quad \text{in } (-),$$

with $F_\pm(-z) = F_\pm(z)$ if the body is symmetrical about $z = 0$, may be handled in precisely the same way as the flat plate, resulting in

$$w_\pm = \frac{z - \zeta(t)}{\frac{l_1}{2} - F_\pm(z)}, \tag{3.6}$$

where $\zeta(t)$ is to be determined by the conditions

$$p_+(\pm h) = p_-(\pm h). \tag{3.7}$$

However, if $S(z) = 1 - \beta z$ here as well, then $\zeta \equiv 0$ as for the flat plate, simply by symmetry. Eventually one finds

$$\rho = 1 - \beta z \quad (|z| > h), \tag{3.8a}$$

$$\rho_+ = 1 - \beta\eta_+\left(\int_0^z F_+(\lambda)\,d\lambda - z l_1\right) \quad (|z| < h), \tag{3.8b}$$

$$\rho_- = 1 - \beta\eta_-\left(-\int_0^z F_-(\lambda)\,d\lambda + z l_2\right) \quad (|z| < z_0), \tag{3.8c}$$

where $z_0 = z_0(t)$ and is given implicitly by

$$l_2 z_0(t) = L_2 h - \int_{z_0}^h F_-(\lambda)\,d\lambda. \tag{3.8d}$$

By the same reasoning as before,

$$\hat{\rho}_- = 1-\beta\eta_+\left(l_2 z \mp (L_1+L_2)\,h \pm \int_0^h F_+(\lambda)\,d\lambda + \int_z^h F_- d\lambda\right) \quad (z_0 < \pm z < h). \quad (3.8e)$$

The functions $\eta_+(\xi)$ and $\eta_-(\xi)$ are solutions $z = \eta_\pm(\xi)$ of

$$L_{\substack{1\\2}} z - \int_0^z F_\pm(\lambda)\,d\lambda = \mp\xi \quad (3.8f)$$

respectively, and, as before,

$$\frac{\partial\hat{\rho}_-}{\partial z} = \beta\frac{l_2 - F_-(z)}{L_1 - F_+((1-\hat{\rho}_-)/\beta)} > 0.$$

This solution can be extended further to the case of an arbitrarily shaped body in an arbitrary density gradient, but the algebraic complexity becomes formidable.

4. The drag, $\kappa \equiv 0$

Contrary to what happens in the case of a plate rising in a rotating fluid, the leading term in the drag may be calculated without appealing to the boundary-layer structure. In fact,

$$D = \int_{-h}^{h}\left(\frac{p_+(z)}{(1+[F'_+(z)]^2)^{\frac{1}{2}}} - \frac{p_-(z)}{(1+[F'_-(z)]^2)^{\frac{1}{2}}}\right)dz, \quad (4.1)$$

which may be integrated by parts once, using the equation of hydrostatic balance and (3.7), to give

$$D = 2g\int_0^h [\rho_+(z)f_+(z) - \rho_-(z)f_-(z)]\,dz, \quad (4.2)$$

with the geometry contained in

$$f_\pm = \int_0^z \frac{d\lambda}{(1+[F'_\pm(\lambda)]^2)^{\frac{1}{2}}}. \quad (4.3)$$

It should be noted that the drag is hydrostatic, depending on the instantaneous density distribution, and is independent of the instantaneous velocity of the body. Thus, if the body is brought to rest, the force remains.

The drag is quite easily found from equations (3.5) to be

$$D_1 = \tfrac{1}{3}\beta g h^3\left(\frac{[L_1+L_2]}{L_1}\right)(1-[L_2/l_2]^2). \quad (4.4)$$

We recall, however, that $\hat{\rho}_-(z) > 0$ in $z_0 < |z| < h$, and we anticipate that some mixing will take place behind the body, leading to a redistribution of density. There are several possibilities. Suppose first that the two strips $z_0 < |z| < h$ mix to a uniform density; then it follows from the conservation of mass in the mixing process that $\hat{\rho}_-(z)$ in (3.5*c*) and (3.5*d*) must be replaced by

$$\rho_2^* = 1 - \tfrac{1}{2}\beta h(1+l_1/L_1)\,\mathrm{sgn}\,z, \quad (4.5)$$

in which case the drag increases to D_2,

$$D_2 = D_1 + \tfrac{1}{6}\beta g h^3(L_1-l_1)^3/L_1 l_2^2. \quad (4.6)$$

However, this new density profile shown in figure 2 (*b*) is such that $\rho_2^*(z_0) > \rho_-(z_0)$, so heavier fluid is still over lighter fluid. Suppose, then, that the mixing process penetrates into the stably stratified fluid in $(-)$ to an additional depth δz_0, such that $\hat{\rho}_-$ is replaced by

$$\rho_3^* = \rho_-(z_0(1-\delta))$$

to give the profile in figure 2 (*c*). Then the density profile is uniformly stable (and continuous) in $(-)$. Again from the conservation of mass, one can show that

$$\delta = \frac{l_2-L_2}{L_2}\left(\left[1+\frac{L_2}{L_1}\right]^{\frac{1}{2}}-1\right)$$

and the increased drag is now

$$D_3 = D_2+\tfrac{1}{3}(D_2-D_1)\left[2\frac{L_1+L_2}{L_2}\left(\left[1+\frac{L_2}{L_1}\right]^{\frac{1}{2}}-1\right)-1\right]. \tag{4.7}$$

Suppose finally that the mixing process could be so effective that *all* of the fluid in $(-)$ mixes, as in figure 2 (*d*). Then, $\rho_- \equiv 1$, and the drag is

$$D_4 = -\tfrac{2}{3}\beta g h^3, \tag{4.8}$$

which is at first sight a curious result. Similar expressions may in principle be obtained for any $F_\pm(z)$.

However, case (*d*) can be ruled out from considerations of the potential energy of the fluid behind the plate. It is easy to show that

$$V_d > V_a > V_b > V_c,$$

where V_d denotes the potential energy with the profile of figure 2 (*d*), and so on. Now, if the Froude number is small, the kinetic energy will be small compared with the potential energy and there is no mechanism for the potential energy to be increased by mixing. Thus, case (*d*) is impossible. Further mixing of profile (*c*) clearly increases the potential energy, and this case is therefore to be preferred on the grounds that mixing will cease when the potential energy is a minimum.

For the density distribution of case (*a*), it is easily proved that the rate of increase of potential energy is equal to the rate of working by the drag force. For cases (*b*) and (*c*), these rates are not equal; the difference going into kinetic energy of the mixing process which is assumed dissipated by turbulence.

5. The boundary layers for $\kappa \equiv 0$

The solution just given violates no-slip boundary conditions on the solid boundaries. It is easy to see that the boundary-layer equations for the end wall $x = L_1$ are

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z} = 0,$$

$$[\rho-\rho_+(z)]\,g = \nu\frac{\partial^2 w}{\partial x^2}, \tag{5.1}$$

$$\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+w\frac{\partial \rho}{\partial z} = 0,$$

which are to be solved for the boundary conditions

$$u = w = 0 \quad \text{on} \quad x = L_1,$$

$$w \to \frac{Uz}{l_1}, \quad \rho \to 1 - \frac{\beta z l_1}{L_1} \quad \text{as} \quad x - L_1 \to -\infty. \tag{5.2}$$

The equations are non-linear and the structure is unsteady; simple solutions do not appear to exist. The thickness of the boundary layer is easily estimated, from the facts that $w \sim Uh/L_1$ and $\rho - \rho_+ \sim \beta h$, to be

$$\delta = \left(\frac{\nu U}{\beta g L}\right)^{\frac{1}{2}} = A^{\frac{1}{2}} h^{\frac{3}{2}} L^{-\frac{1}{2}}. \tag{5.3}$$

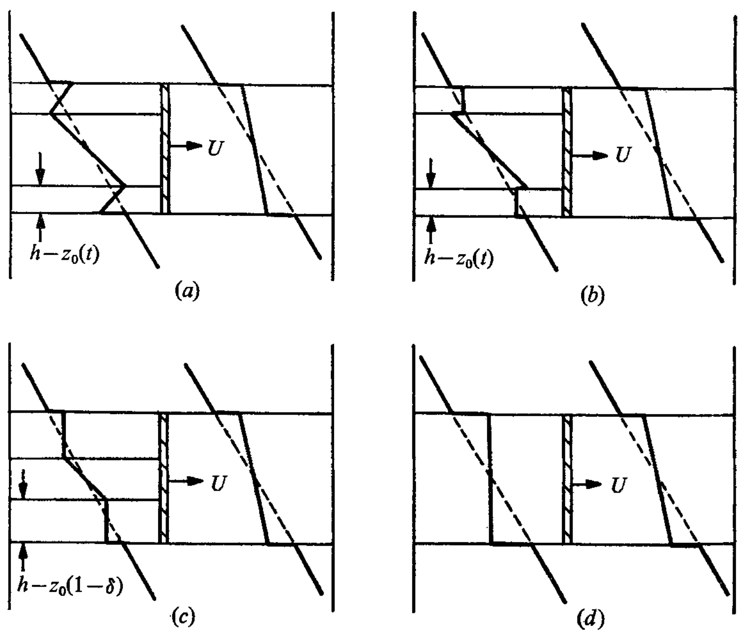

FIGURE 2. Possible density profiles under various assumptions about the mixing process for the flat plate. (*a*) No mixing. (*b*) Mixing of unstable layers. (*c*) Mixing penetration into stable region far enough to make profile stable. (*d*) Mixing of all fluid behind the plate.

Clearly, a layer of this kind can exist only if the diffusion term is negligible, i.e.

$$\kappa/\delta^2 \ll U/L \quad \text{or} \quad Pe \gg (h/\delta)^2 = L/hA. \tag{5.4}$$

This is a more stringent condition than just $Pe \gg 1$.

6. The horizontal shear layers for $\kappa \equiv 0$

On $z = \pm h$, the solutions just given exhibit discontinuities. In particular, on $z = h$,

$$[w_\pm] = \mp hU/l_{\substack{1\\2}}, \tag{6.1}$$

$$[u_\pm] = -l_{\substack{1\\2}}/L_{\substack{1\\2}}\,U. \tag{6.2}$$

In a layer of thickness δ, mass conservation and equation (6.1) require a horizontal velocity of order Uh/δ; however, equation (6.2) requires only $u = O(U)$.

This layer cannot be hydrostatic, because then there is no pressure gradient to move fluid from $(+)$ to $(-)$. Apart from mass conservation, the most general shear-layer equations are

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0, \tag{6.3a}$$

$$\frac{\partial p}{\partial x}=\nu\frac{\partial^2 u}{\partial z^2}, \tag{6.3b}$$

$$\frac{\partial p}{\partial z}+\rho g=0. \tag{6.3c}$$

Because the large horizontal velocity reduces variations along the layer, it is appropriate to write

$$\rho=\rho_0(z,t)+\rho'((z-h)/\delta,x,t),$$

where $\rho'=o(\rho_0)$ and ρ' vanishes at the edge of such a layer for matching. The mass conservation equation is then

$$u\frac{\partial \rho'}{\partial x}+w\frac{\partial \rho'}{\partial z}=(w_0(h,t)-w)\left.\frac{\partial \rho_0}{\partial z}\right|_h, \tag{6.3d}$$

where w_0 is the vertical velocity corresponding to ρ_0. From (6.1), $w=O(w_0)$, so from this equation $\rho'\sim\beta\delta$. Equations (6.3a)–(6.3c) give $\rho'\sim\nu UhL/g\delta^4$, hence

$$\delta=(UhL\nu/g\beta)^{\frac{1}{5}}=(Lh^4)^{\frac{1}{5}}A^{\frac{1}{5}}. \tag{6.4}$$

Now w_0 and ρ_{0z} are different for $z\rightarrow h-$ and $z\rightarrow h+$, so there must be three distinct layers on h: $z<h$, $x<Ut$ and $z<h$, $x>Ut$ and $z>h$. Written in the usual boundary-layer variables, with $\zeta\equiv(z-h)/\delta$, (6.3) is

$$L\frac{\partial \tilde{u}}{\partial x}+\frac{\partial \tilde{w}}{\partial \zeta}=0,$$

$$\frac{\partial^3 \tilde{u}}{\partial \zeta^3}+L\frac{\partial \tilde{\rho}}{\partial x}=0,$$

$$L\tilde{u}\frac{\partial \tilde{\rho}}{\partial x}+\tilde{w}\frac{\partial \tilde{\rho}}{\partial \zeta}+k\left(\frac{w_0(h)\,L}{Uh}-\tilde{w}\right)=0. \tag{6.5}$$

Here, $k=\pm l_{\frac{1}{2}}/L$, $w_0L/Uh=\pm L/l_{\frac{1}{2}}$ for the layers on $z=h-$ and $k=1$, $w_0\equiv 0$ on $z=h+$.

The solutions to (6.5) are not easily found because of their non-linear character. For diffusion of buoyancy to be negligible, we need

$$\kappa\ll U\delta h/L \quad \text{or} \quad Pe\gg(h/L)^{\frac{1}{5}}A^{-\frac{1}{5}}. \tag{6.6}$$

The neglect of horizontal momentum convection in (6.3b) requires

$$\epsilon\ll(L/h)^{\frac{2}{5}}A^{\frac{2}{5}}. \tag{6.7}$$

There is a density adjustment layer of relatively simple structure sandwiched between these shear layers if κ is small but non-zero; its thickness is $(\kappa^5\nu L^6/g\beta U^4h^4)^{\frac{1}{10}}$. Details of this interior layer and the asymptotic structure of solutions to (6.5) as $\zeta\rightarrow-\infty$ may be found in Foster (1969).

7. Small Peclet number solution

If the Prandtl number is order one, then $Pe = \nu\epsilon^2 h/\kappa AL$ shows that

$$\epsilon \ll (L/h)^{\frac{1}{2}} A^{\frac{1}{2}}$$

is equivalent to $Pe \ll 1$. If $Pe \ll 1$, equation (1.2) shows that the convection of buoyancy is unimportant. In that case,

$$\partial\rho/\partial t = \kappa\nabla^2\rho \tag{7.1}$$

is the appropriate equation to be solved with (1.7). Now, if the moving body is again the plate $x = Ut$, $|z| < h$, and the boundary condition is $\rho_x = 0$ on solid surfaces, then the exact solution for all t is

$$\rho = 1-\beta z. \tag{7.2}$$

We write
$$\rho = 1-\beta z+\bar{\rho},$$
$$p = -g(z-\tfrac{1}{2}\beta z^2)+\bar{p},$$

and substitute into (1.1)–(1.3), the result being

$$\nabla . \mathbf{u} = 0, \tag{7.3a}$$

$$\frac{D\bar{\rho}}{Dt}-\beta w = \kappa\nabla^2\bar{\rho}, \tag{7.3b}$$

$$\frac{D\mathbf{u}}{Dt}+\nabla\bar{p} = -\bar{\rho}g\mathbf{k}+\nu\nabla^2\mathbf{u}. \tag{7.3c}$$

The boundary conditions are given by (1.4) and (1.5) with $\bar{\rho}$ for ρ, and the initial condition is $\bar{\rho} = 0$ on $t = 0$. As before, convection of $\bar{\rho}$ is small compared with diffusion if $Pe \ll 1$. In addition, if the convection term in (7.3*c*) is to be dominated by buoyancy forces, we need $U \ll g\beta h^2 L/\kappa$ or

$$Pe \ll (\nu/\kappa)\, Ra. \tag{7.4}$$

Here, Ra is the Rayleigh number $g\beta h^4/\kappa\nu$. Hence, under $Pe \ll 1$ and (7.4), equations (7.3) are

$$\nabla . \mathbf{u} = 0, \tag{7.5a}$$

$$-\beta w = \kappa\nabla^2\bar{\rho}, \tag{7.5b}$$

$$\nabla\bar{p}+\bar{\rho}g\mathbf{k} = \nu\nabla^2\mathbf{u}. \tag{7.5c}$$

The viscous term will be negligible outside the boundary and shear layers if $Ra \gg 1$, which we now take to be true.

In the equivalent rotating-fluid problem, the structure of the entire flow depends strongly on the boundary layers. Equations (7.5) show such a vertical boundary layer to have thickness $(\kappa\nu/\beta g)^{\frac{1}{4}}$; if $\xi \equiv (\beta g/4\kappa\nu)^{\frac{1}{4}}(x-x_B)$ the boundary-layer equations on $x = x_B$ are

$$\left.\begin{aligned}\frac{1}{2}\frac{\partial^2(\bar{\rho}-\bar{\rho}_0)}{\partial\xi^2} &= \left(\frac{\beta}{g}\right)^{\frac{1}{2}}(w_0-w),\\ \frac{1}{2}\frac{\partial^2 w}{\partial\xi^2} &= \left(\frac{g}{\beta}\right)^{\frac{1}{2}}(\bar{\rho}-\bar{\rho}_0),\end{aligned}\right\} \tag{7.6}$$

where $(\ )_0$ denotes the solution in the interior. Integrating the first of these across the layer $x > x_B$ say,

$$\left(\frac{\beta}{g}\right)^{\frac{1}{2}}\int_0^\infty (w-w_0)\,d\xi = -\left(\frac{4\kappa\nu}{\beta g}\right)^{\frac{1}{4}}\frac{\partial\bar{\rho}_0}{\partial x}\bigg|_{x=x_B}, \tag{7.7}$$

where (1.4) and (1.5) have been used. The ratio of the right-hand side to the left-hand side is $(h/L)\,(\nu/\kappa)^{\frac{1}{2}}\,Ra^{-\frac{3}{4}}$, so, to leading order, such a layer can carry no mass; thus, in particular, it cannot carry fluid from $(+)$ to $(-)$ as in the rotating-flow problem. Therefore, w is order Uh/L in the interior of $(+)$ and $(-)$.

So, the interior problem involves solution of

$$\frac{\partial u_0}{\partial x}+\frac{\partial w_0}{\partial z} = 0, \tag{7.8a}$$

$$-\beta w_0 = \kappa\nabla^2\bar{\rho}_0, \tag{7.8b}$$

$$\nabla\bar{p}_0+\bar{\rho}_0 g\mathbf{k} = 0. \tag{7.8c}$$

Since (7.8a) and (7.8c) are the same as (3.1a) and (3.1c), the velocity field is identical and given by (3.2), (3.3). Then (7.8b) may be integrated to give

$$\bar{\rho}_{0\pm} = \pm\frac{1}{6}\frac{\beta U}{\kappa l_{\frac{1}{2}}}z^3+a_\pm z+b_\pm \quad \text{in } (\pm),$$

$$\bar{\rho}_0 \equiv 0, \quad |z| > h.$$

Diffusion precludes a discontinuity in $\bar{\rho}$, so $\rho_\pm = 0$ on $|z| = h$ gives the unique result

$$\bar{\rho}_{0\pm} = \mp\frac{\beta U z}{6\kappa l_{\frac{1}{2}}}(z^2-h^2). \tag{7.9}$$

Notice that the buoyancy flux leaving $(+)$ enters $(-)$, i.e.

$$l_1\frac{\partial\bar{\rho}_{0+}}{\partial z}+l_2\frac{\partial\bar{\rho}_{0-}}{\partial z} = 0 \quad \text{on} \quad |z| = h,$$

so there *does* exist a jump in ρ_z on $|z| = h$, viz.

$$\left[\frac{\partial\bar{\rho}_{0\pm}}{\partial z}\right] = \pm\frac{\beta U h^2}{3\kappa l_{\frac{1}{2}}}, \tag{7.10}$$

which necessitates a shear layer for its removal, as do the velocity jumps $[w]$ and $[u]$.

Having completed the interior solution, the solution to (7.6) on $x = x_{B\pm}$ is

$$w = w_0(1-e^{\mp\xi}\cos\xi)-\left(\left(\frac{4\kappa\nu g}{\beta^3}\right)^{\frac{1}{4}}\frac{\partial\bar{\rho}_0}{\partial x}\bigg|_{x_B}\mp w_0\right)e^{\mp\xi}\sin\xi. \tag{7.11}$$

8. The drag, $Pe \ll 1$

The drag is easily obtained from (4.2), since $\bar{\rho}_0$ is still hydrostatic,

$$D = \frac{2g\beta U h^5}{45\kappa}\left(\frac{1}{l_1}+\frac{1}{l_2}\right),$$

which is of order $g\beta h^3 Pe$ and, as such, is very small compared with the $Pe \gg 1$ drag. Notice that (7.9) indicates $|\bar{\rho}_z| \ll \beta$, so the density gradient always remains stable, and the mixing modifications of §4 are not required. Further, the drag depends on the instantaneous value of U and in this case disappears if the body is brought to rest.

9. The horizontal shear layers

Equation (2.7) showed that the vertical boundary layers could carry no mass to leading order. Using conservation of mass, (7.7) may be written in the form

$$u_B - u + \frac{\kappa}{\beta}\frac{\partial^2 \overline{\rho}}{\partial x\,\partial z} = 0 \qquad (9.1)$$

on vertical solid boundaries. We neglected the viscous term in (7.5*c*) to compute the interior solution of §7. Therefore, to remove $[\overline{\rho}_z]$, $[w]$ and $[u]$, we retain the viscosity. Certainly (7.5*a*) and (7.5*b*) must be of the same form as in the interior, i.e. (7.8*a*) and (7.8*b*),

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \qquad (9.2a)$$

$$-\beta w = \kappa\,\partial^2\overline{\rho}/\partial z^2. \qquad (9.2b)$$

If the layer is thin, then $w \ll u$, so that (7.5*c*) is

$$\frac{\partial \overline{p}}{\partial x} = \nu\frac{\partial^2 u}{\partial z^2}, \qquad (9.3a)$$

$$\partial\overline{p}/\partial z = -g\overline{\rho}, \qquad (9.3b)$$

which may be reduced, by eliminating $\overline{p}$, to

$$\nu\frac{\partial^3 u}{\partial z^3} + g\frac{\partial\overline{\rho}}{\partial x} = 0. \qquad (9.4)$$

By using (9.2) and (9.4), the layer thickness δ is easily found to be $(\kappa\nu L^2/g\beta)^{\frac{1}{6}}$. If $[\overline{\rho}_z]$ is to be removed, then $\overline{\rho}$ in this shear layer must be of order $Pe\,\delta\beta$. From (9.4), then, $u \sim (g\beta\kappa/\nu)^{\frac{1}{2}}\,\delta Pe$; the u corresponding to that required to remove $[w]$ is order $(g\beta\kappa/\nu)^{\frac{1}{2}}\,\delta^2 Pe/h$; to remove $[u]$ requires $u \sim (g\beta\kappa/\nu)^{\frac{1}{2}}\,\delta^3 Pe/h^2$. Therefore, the leading-order problem for the shear layer is $u \sim (g\beta\kappa/\nu)^{\frac{1}{2}}\,\delta Pe$, $\overline{\rho} \sim Pe\,\delta\beta$. Scaling u, $\overline{\rho}$, and w appropriately and $\zeta \equiv (z - z_0)/\delta$, where z_0 is the line on which the discontinuity exists, then give, for (9.2) and (9.4),

$$L\frac{\partial u^*}{\partial x} + \frac{\partial w^*}{\partial \zeta} = 0, \qquad (9.5a)$$

$$\frac{\partial^2\rho^*}{\partial\zeta^2} + w^* = \left(\frac{L}{h}\right)^{\frac{1}{3}} Ra^{-\frac{1}{6}}\,\frac{w_0(z_0)\,L}{Uh}, \qquad (9.5b)$$

$$\frac{\partial^3 u^*}{\partial\zeta^3} + L\frac{\partial\rho^*}{\partial x} = 0, \qquad (9.5c)$$

and (9.1) gives

$$u^* = Ra^{-\frac{1}{3}}\left(\frac{L}{h}\right)^{\frac{2}{3}}\frac{u_B}{U} + \frac{h^{\frac{4}{3}}Ra^{-\frac{1}{3}}}{L^{\frac{1}{3}}}\,\frac{\partial^2\rho^*}{\partial\zeta\,\partial x} = 0 \qquad (9.5d)$$

on vertical boundaries.

This problem may be solved by expanding ( )* in powers of $Ra^{-\frac{1}{6}}(L/h)^{\frac{1}{3}}$. The details of the solutions are nearly identical with the '$\frac{1}{3}$ layer' solutions given by Moore & Saffman (1969) in connexion with the shear layers for a body rising through a rotating fluid, and so are not included here. The jumps $[\bar{\rho}_z]$, $[w]$, and $[u]$ are removed by respectively higher-order solutions in the perturbation expansion.

10. Summary

We found that the drag made non-dimensional with $\beta g h^3$ is proportional to Pe for $Pe \ll 1$, but is order unity as $Pe \to \infty$, provided the motion is sufficiently slow in the sense previously described.

In the $Pe \gg 1$ analysis, $\epsilon \ll 1$, $A \ll 1$ were sufficient to assure the validity of the drag calculation. That diffusion be negligible in the vertical boundary layers required further that

$$Pe \gg L/Ah.$$

If the shear layers are to have the structure suggested in §6, the neglect of convection of horizontal momentum requires

$$\epsilon \ll (L/h)^{\frac{2}{5}} A^{\frac{2}{5}}.$$

Since equations (6.3) are non-linear as they stand, one might as well retain the complete horizontal momentum equation, and hence require only $\epsilon \ll 1$.

Since $\nu \sim \kappa$ in the $Pe \ll 1$ solution, only one other parameter must be specified, $Ra \gg 1$. It appears that there are no restrictions on the solution more severe than these.

The similarities with the rotating-fluid problem mentioned before are minimal, mainly because the boundary conditions are kinematic. A Taylor column does indeed exist, but its character and the associated shear and boundary-layer structure are radically different.

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